LOGICAL IMPLICATION AS THE OBJECT OF MATHEMATICAL INDUCTION

Anderson Norton
Virginia Tech
norton3@vt.edu

Rachel Arnold
Virginia Tech
rlongley@vt.edu

Proof by mathematical induction poses persistent challenges for college mathematics students. We use an action-object framework to analyze ways that students might overcome these challenges. We conducted three pairs of interviews with students enrolled in a proofs course. Tasks were designed to elicit student understanding of logical implication and components of proof by induction. We report results from one student, Mike, who had constructed logical implication as an object, and who invented a quasi-inductive proof.

Keywords: Logical Implication, Mathematical Induction, Mathematical Objects, Reasoning and Proof

Mathematical induction relies on two defining properties of the natural numbers: 1 is a natural number; and if k is a natural number, then k+1 is also a natural number. Any set with these two properties contains the natural numbers. In particular, if we define S as the set of natural numbers n for which an open proposition P(n) holds, then we can show S=N by showing the following: (1) P(0) is true, and (2) for any natural number k, if P(k) is true then P(k+1) is also true, written P(k)→P(k+1). In other words, P(n) holds for all natural numbers n if S satisfies the two defining properties of the natural numbers. The inductive implication P(k)→P(k+1) can be treated in one of two ways: as an inductive step from the inductive assumption, P(k); or, as an invariant relationship between P(k) and P(k+1) for any k. We apply an action-object framework to the study of logical implication and its use in proofs by induction.

Action-Object Theory

Piaget (1970) distinguished logico-mathematical knowledge from other forms of knowledge via its objects of study and, specifically, how they are created. Following Piaget, we define mathematical objects as coordinated mental actions. Our distinction between logical implication as a transformation and logical implication as an invariant relationship builds upon Piaget's action-object theory of mathematical development. Dubinsky made a similar distinction between actions and objects in APOS theory, which is also derived from Piaget's genetic epistemology (Dubinsky & McDonald, 2001). Within that framework, Dubinsky (1986) conjectured that understanding implication as an object could empower students in mastering proof by induction. Our study is an investigation of this claim within our own action-object framework.

Methods

The first author conducted clinical interviews with students from an Introduction to Proofs course taught by the second author. The course is a junior-level mathematics course designed to prepare mathematics majors for rigorous expectations in subsequent proofs-based courses. Three students volunteered for the study, and all three students were invited to participate in a pair of clinical interviews—one interview before mathematical induction was taught in class and one after. All of the interviews were video-recorded, and they lasted about 45 minutes. Each interview consisted of the students responding to tasks that were designed to elicit their reasoning and understanding. These tasks consisted of three types: logical implication, components of mathematical induction, and formal proof by induction (see Table 1).

Table 1: Sample tasks.

Task Type	**Sample**
A. Logical Implication	Suppose the statement is true: "If two topological spaces are homeomorphic, their homology groups are isomorphic." Evaluate whether the following statements are **true**, **false**, or **uncertain**. 1) [converse] 2) [contrapositive] 3) [negation]
B. Components of Mathematical Induction	Suppose P(n) is a statement about a positive integer n, and we want to prove that P(n) is true for all positive integers n. For each scenario, decide whether the given information is enough to prove P(n) *without induction*, *with induction*, or whether the given information is *not enough*. P(1) is true; there is an integer k≥1 such that P(k) implies P(k+1). P(1) is true; for all integers k≥1, P(k) implies P(k+1).
C. Formal Proof by Induction	Prove the following claim: For every positive integer n, $2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$.

Results

We report results for a student named Mike from his first interview. We focus on Mike because he seemed to possess an understanding of logical implication as an object. Mike quickly recognized the three transformed statements in Task A1 as the converse, contrapositive, and negation of the original statement. For example, in evaluating Task A3, he responded, "if the first statement is true, then that has to be false." When asked to justify his response, Mike replied, "it's the negation of the implication." He seemed to recognize how the original statement had been transformed into the three statements he was evaluating, and he did not have to rely on any formal notation in order to do so. We characterize this recognition as an assimilation of the statements within a single structure for comparing and transforming them.

In contrast, Mike did not seem to readily assimilate the scenarios in Task B1. However, he could make critical distinctions between them, particularly with regard to quantifiers like "there exists" and "for all." For example, consider his comparison of Tasks B1 and B3. Note that "M" refers to Mike, and "R" refers to the researcher (first author).

M: [reads Task B1 aloud, then pauses] That's interesting. I don't *think* it's enough because that's only the truth of two, unless you are meant to assume to know that that means that you could like replace that k+1 with some other integer--you know, j--that was… and then j+1 is also true, and so on. But I think it's just not enough.
R: Okay, so what *would* you know from this one?
M: I would know that there's at least... P(1) is true; P(k) and then P(k+1) is true. And that's all I really know, from this.
R: So you would know P(1) is true, P(k) is true, and P(k+1) is true.
M: Yeah.
R: But that's not enough to prove…?
M: That's not enough to prove all positive integers.
R: So, you know P(k) is true, for which k's?
M: Just, there is a k. Just one k.
R: Is that the problem, then?
M: I think that's the problem.
R: [shows Task B2 on paper]
M: [looks at statement and immediately responds] Yeah, I know this is what, this is what it

would take. P(1) is true and for all integers, k, greater than 1, P(k) implies k+1 is true. So, that is the like sort of recursive thing that I was talking about, up there, where like… If you know that, yeah, so… P(1) is true and for all integers greater than 1, P(k) implies P(k+1), so that just grows to encompass all reals.

R: How does this recursive thing work?

M: Well, recursive is the wrong word. I mean, it just like… If you know P(k). Let's say k is equal to 2. So then P(2) implies P(3). But since it's for all integers k greater than 1, then P(3) also implies P(4), and so on.

Despite having never seen proof by induction, Mike was able to engage in the cognitively demanding task of analyzing its components in Task B. His objectification of logical implication seemed to free up his cognitive resources for focusing on *where,* not *how,* to apply the inductive implication. However, Mike's struggle with the increased demands of Task B were evidenced by his numerous pauses and minor mis-statements (e.g., "k greater than 1").

Upon reading Task B2, Mike immediately recognized what had been missing in Task B1. The new scenario allowed the implication to be applied to all values greater than or equal to 1. Mike began to recursively apply the logical implication, $P(k) \rightarrow P(k+1)$, in a manner consistent with what Harel (2002) called "quasi-induction." In both Tasks B1 and B2, Mike was eager to successively apply the inductive implication to consecutive pairs of integers. After sorting through the cognitive demands of the tasks, he was able to do so correctly.

Mike subsequently applied his quasi-inductive reasoning to Task C. Although the task did not explicitly call for induction, Mike independently attempted to prove the claim that way.

M: Well, I'm sort of thinking here that like, start with n equals 1 because you know, the simplest to add all of them up. So, you have 2 is equal to 2 to the second minus 2, just… And I was thinking, if you could write it in sort of like a symbolic way where you have like the next one where n equals 2, then you have 2 plus 2 squared is equal to 8 minus 2. And then I was thinking maybe you could plug like the 2 squared minus 2 in for this initial 2 and get it to like [moves right hand in circular motion], you know, build on itself.

R: Oh. Okay. Um, I think that's a good idea. Um, does that relate to any of [the Task B scenarios]...

M: Yeah. Yeah, it does. That's what sort of gave me the idea... is like for the k+1.

R: Okay. These gave you the idea for doing that? Which of these scenarios would it best fit?

M: Well, hopefully [Task B2].

In Task C, P(n) is the statement "$2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$." Mike's approach was to use P(1) to build the equation in P(2) by substituting the right hand side (2^2-2) of the equation of P(1) for the initial 2 in the left hand expression in P(2). In this way, Mike used an inductive approach to show $P(1) \rightarrow P(2)$. Mike did not complete a formal proof by induction because he did not know how to write the inductive implication in a "symbolic way" that would generalize from any case, k, to the next case, k+1. However, Mike was conceivably on his way to generalizing his quasi-inductive argument into a formal proof by mathematical induction.

Conclusions

Piaget (1970) characterized logico-mathematical thought as grounded in composable and reversible mental actions. He described mathematical objects as coordinations of such actions. Dubinsky (1986) conjectured that treating logical implication as an object enables students to reason in more powerful ways, specifically with constructing proofs by mathematical induction. We investigated and affirmed Dubinsky's conjecture through our interviews with college mathematics students, like Mike.

Mike entered our study with an understanding of logical implication as an object. He coordinated actions on logical implications as objects to organize components of mathematical induction into inductive arguments. He used implication across particular pairs of cases (e.g. P(2) implies P(3)) in a manner that fits Harel's (2002) description of quasi-induction. Mike's struggles were limited to the following: (1) determining the cases in which the object applied; (2) symbolizing the inductive implication in a way that generalized to all valid cases.

Mike's first struggle relates to the role of (hidden) quantifiers and students' difficulties in differentiating between "there exist" and "for all" statements (Shipman, 2016). He recognized that Task B1 was existentially quantified, but struggled with the difference between an arbitrary variable and a fixed, unknown value. However, because Mike had logical implication as a mental object, he was able to resolve the details of the quantification. Mike's second struggle was apparent in his pre-interview in that he could not symbolically state the inductive implication for an arbitrary k. However, by the post-interview, he easily formalized a general inductive implication, possibly due to instruction on mathematical induction.

Mike did not seem to experience difficulty with other common challenges reported in prior research on mathematical induction. For example, he did not conflate the inductive assumption with assuming the proposition he was supposed to prove (cf. Avital & Libeskind, 1978; Ron & Dreyfus, 2004). We argue that students like Mike, who understand logical implication as an object, avoid this pitfall of conflation by treating the inductive assumption as a component of a larger object. For them, the inductive assumption is not an independent claim, rather it exists within the implication that must be established.

Piagetian theory offers a lens through which Dubinsky (1986) identified the objectification of logical implication as a potentially critical aspect of mastering proof by mathematical induction. Our results suggest ways that instruction can build upon such understanding. One such approach is the task sequence used in our study (Task A, Task B, and then Task C), which seemed to guide Mike to nearly invent mathematical induction. His independent formulation resembled Harel's (2002) quasi-induction, which Harel had recommended as an instructional approach. Instructional methods that separate the inductive hypothesis from the inductive step may inadvertently discourage students from engaging in quasi-inductive reasoning.

References

Avital, S., & Libeskind, S. (1978). Mathematical induction in the classroom: Didactical and mathematical issues. *Educational Studies in Mathematics*, *9*(4), 429-438.

Dubinsky, E. (1986). Teaching mathematical induction: I. *The Journal of Mathematical Behavior*.

Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In *The teaching and learning of mathematics at university level* (pp. 275-282). Springer Netherlands.

Harel, G. (2002). The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based Instruction'23. *Learning and teaching number theory: Research in cognition and instruction*, *2*, 185.

Piaget, J. (1970). *Structuralism* (C. Maschler, Trans.). New York: Basic Books.

Ron, G., & Dreyfus, T. (2004). The use of models in teaching proof by mathematical induction. In *Proceedings of the 28th Conference of the International* (Vol. 4, pp. 113-120).

Shipman, B. A. (2016). Subtleties of hidden quantifiers in implication.*Teaching Mathematics and its Applications*, *35*(1), 41-49.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.